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Ergodicity of a polling network

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The polling network considered here consists of a finite collection of stations visited successively by a single server who is following a Markovian routing scheme. At every visit of a station a positive random number of the customers present at the start of the visit are served, whereupon the server takes a positive random time to walk to the station to be visited next. The network receives arrivals of customer groups at Poisson instants, and all customers wait until served, whereupon they depart from the network. Necessary and sufficient conditions are derived for the server to be able to cope with the traffic. For the proof a multidimensional imbedded Markov chain is studied.

queuing networks * polling network * multidimensional Markov chain * ergodicity

1. Introduction and model

The present paper is devoted to the question how much traffic a certain polling system can cope with. Polling systems are used for modelling technical systems in which a fixed number of ‘servers’ are polling a given collection of ‘users’ in deterministic or random order and are providing a certain amount of service every time they find a user demanding it. Communication, computer, production and road-traffic networks provide examples. For a survey of the extensive literature see [7].

The special system to be considered is a basic one for this area. There are N users (or stations), numbered 1 through N, and a single server is polling them in Markovian fashion: given the server has been polling station i the next one to be polled is station j with probability r_{ij}, where the ‘routing’ matrix (r_{ij}) is assumed to be irreducible with stationary distribution (π_i). The entire system of stations is fed by an external arrival stream of customers. At the instants of a Poisson stream with a fixed intensity a group of customers joins each of the stations, and it is assumed that the joint group sizes are i.i.d. and are such that a positive

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average of λ_i customers join station i per unit of time. All customers wait until they have received a service in the station at which they arrived and they depart from the system at once thereafter. Whenever the server arrives at station i there will be a number of consecutive services, distributed as $\min(x_i, D_i)$, where x_i is the number of customers present at station i at the instant of the server's arrival, and where D_i is an arbitrarily distributed random variable on the positive integers with finite mean d_i, $i=1, \ldots, N$. The service times in station i are drawn independently from an arbitrary distribution on the positive real numbers with finite mean b_i, $i=1, \ldots, N$. Given a switch from station i to station j, the server will spend a finite time to 'walk' from i to j, distributed arbitrarily on the positive real numbers with finite (positive) mean w_{ij}. Arrival stream, routing, walking, and services are mutually independent. Let $S(n)$ denote the nth station polled by the server, and let $X_i(n)$ denote the number of customers present at station i at the moment the server finishes the walk from the $(n-1)$st to the nth station polled. Then, due to the assumptions made above, the stochastic process $(S, X) = \{(S(n), X_1(n), \ldots, X_N(n)); n \geqslant 0\}$ is an irreducible Markov chain, and the 'stability' question posed at the outset is answered by the following theorem.

Theorem 1.1. *The Markov chain (S, X) is positive recurrent if and only if*

$$\lambda_i \bar{w}_N < (1-\rho_N)\pi_i d_i, \tag{1.1}$$

$i=1, \ldots, N$, *where*

$$\rho_N = \sum_{i=1}^{N} \lambda_i b_i$$

and

$$\bar{w}_N = \sum_{i=1}^{N} \pi_i \sum_{j=1}^{N} r_{ij} w_{ij}.$$

Note that, by assumption on the w_{ij}, the average walking time $\bar{w}_N$ is positive, and hence (1.1) cannot hold unless $\rho_N < 1$. The necessity of (1.1) will be proved in Section 2, the sufficiency in Section 3. A by-product of the proof will be the following result.

Corollary 1.2. *If (S, X) is positive recurrent, with stationary distribution $(\pi(i, x_1, \ldots, x_N))$, then*

$$\sum_{(x_1,\ldots,x_N)} \pi(i, x_1, \ldots, x_N) E[\min(x_i, D_i)] = \lambda_i \frac{\bar{w}_N}{1-\rho_N}, \quad i=1, \ldots, N. \qquad \square \tag{1.2}$$

The left-hand side is the expected number of customers served in station i per polling step, in steady state.

An important way of polling is in cyclic order. For this special case, and under the additional special assumptions that only one customer arrives to the system at Poisson instants and that the D_i are deterministic numbers, a proof of Theorem 1.1 has been given in [4].

After finishing the present work the authors learned of [3], which settles the ergodicity question for a model differing from the present one in that it deals with cyclic service only, and has only one customer at a time in the (Poisson) arrival stream to the system, but allows, on the other hand, a richer variety of service disciplines. However, group arrivals, as featured in the present model, allow an interpretation of the physical process that is not covered in [3]: A group of customers arriving at a station, say station i, can be interpreted, for instance, as one individual bringing a number of units of workload into station i, of which up to D_i are worked off per visit; the service times can be chosen identically equal to one time unit, with the effect that the server will spend a maximum of D_i units of time in the station and will preempt a customer's service upon reaching this limit. This is very close to real implementations.

Our analysis assumes finite means d_i. Formally, if one lets $D_i \equiv \infty$, hence $d_i = \infty$ for all i, then one has what is called gated service (see [7]), and Theorem 1.1 suggests that, in this case, positive recurrence prevails if and only if $\rho_N < 1$, independently of the value of $\bar{w}_N$. For the case of cyclic polling and single arrivals, this has been rigorously proved, for example, in [1], and is also covered in [3].

For the purpose of linking the present work with a certain currently rather active area of research in queuing theory suppose that the N stations are represented as points on the unit circle, located at regular intervals of length $1/(2\pi N)$. Assume that just one customer arrives at each arrival instant, and that $\lambda_i = \lambda/N$, $i = 1, \ldots, N$. Let the service times for all stations be drawn from the same distribution with mean b, and consider a server polling in cyclic order or else a 'drunken' server who chooses the next station to be polled always from the nearest two, giving each an equal chance. In both cases, $\pi_i = 1/N$, $i = 1, \ldots, N$. Let the walking time to the next station be identically equal to w/N for some positive fixed w. Finally, let $D_i \equiv 1$ for all i. Then (1.1) becomes:

$$\frac{\lambda}{N}\frac{w}{N} < (1-\lambda b)\frac{1}{N} \quad \text{or} \quad \frac{\lambda w}{N} < 1 - \lambda b\,.$$

As N tends to infinity one obtains, formally, two variants of a type of 'queuing model on a circle', as studied in [5] and [6]. It has been shown in these papers that the condition $\lambda b < 1$ is necessary and sufficient for the stability of those service systems, and our formal limiting argument gives this, too.

From a mathematical point of view the present paper contains just an 'ergodicity' proof for a type of 'multidimensional' Markov chain. However, proving ergodicity of multidimensional chains is notoriously difficult, and it is believed that the proof technique used here has the potential to bear fruit in other examples of such chains. In fact, an ergodicity proof for a Jackson network, given in [2], is along the same lines.

2. Proof of necessity

This section is devoted to the proof that conditions (1.1) are necessary for the positive recurrence of the Markov chain (S, X) defined in Section 1. For the purposes of Section 3

Theorem 2.1 below will be stated for a class of auxiliary polling systems, one of which is the one defined in Section 1. The latter shall be called system H. An auxiliary polling system, to be denoted by H_{i_0}, differs from H only in the modified rule that, whenever the server polls one of the stations j, $i_0<j\leqslant N$, the number of services performed there is drawn from the distribution of D_j, no matter how many customers, x_j, are present in station j at the time of the server's arrival. Thus the number of customers removed in station j will be equal to $\min(D_j, x_j)$, and if $D_j>x_j$, then there will be D_j-x_j 'dummy' services. Clearly, system H_N is identical with system H.

Throughout this section, system H_{i_0} will be considered for an arbitrary, fixed i_0, and all quantities will be defined with respect to H_{i_0}, without any notational efforts to point to this fact.

For system H_{i_0} let $S(n)$ denote the nth station polled, and let $X_i(n)$, $1\leqslant i\leqslant i_0$, denote the number of customers present at station i at the beginning of the nth visit, $n=0, 1, 2, \ldots$. The sequence $\{(S(n), X(n)); n=0, 1, 2, \ldots\}$, where $X(n)=(X_1(n), \ldots, X_{i_0}(n))^{\mathrm{T}}$, T for 'transposed', is an irreducible Markov chain, also denoted as (S, X). Thus $X(n)$ is viewed as an i_0-dimensional vector, and all further vectors to be introduced below are i_0-dimensional column vectors, likewise. Keep in mind, however, that stations i, $i>i_0$, are still polled in system H_{i_0}. For (S, X) let $\alpha_k(i, j; x)$ denote the expected increment in the number of customers at station k, $1\leqslant k\leqslant i_0$, between two consecutive arrivals of the server at a station, given the first one is station i, the second station j, $1\leqslant i, j\leqslant N$, and given the customer numbers at stations 1 through i_0 at the start of the first visit are given by $\boldsymbol{x}$. Let $\alpha(i, j; \boldsymbol{x})$ denote the corresponding vector. Then

$$\boldsymbol{\alpha}(i, j; \boldsymbol{x})=(w_{ij}+d_i b_i)\boldsymbol{\lambda}, \quad i>i_0,$$

where

$$\boldsymbol{\lambda}=(\lambda_1, \ldots, \lambda_{i_0})^{\mathrm{T}},$$

and

$$\boldsymbol{\alpha}(i, j, \boldsymbol{x})=w_{ij}\boldsymbol{\lambda}-d_i(x_i)\boldsymbol{v}_i, \quad i\leqslant i_0,$$

where $\boldsymbol{x}=(x_1, \ldots, x_{i_0})^{\mathrm{T}}$ and

$$d_i(x_i)=E[\min(x_i, D_i)], \quad \boldsymbol{v}_i=\boldsymbol{e}_i-b_i\boldsymbol{\lambda}, \quad \boldsymbol{e}_i=(0, \ldots, 0, 1, 0, \ldots, 0)^{\mathrm{T}}$$

with the unit component being the ith one, $i=1, \ldots, i_0$.

For the $i_0\times i_0$ matrix V whose ith column vector is $\boldsymbol{v}_i$ $i=1, \ldots, i_0$, the determinant is given by

$$\det V=\det(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{i_0})=\det(\boldsymbol{e}_1-b_1\boldsymbol{\lambda}, \ldots, \boldsymbol{e}_{i_0}-b_i\boldsymbol{\lambda})=1-\sum_{i=1}^{i_0}\lambda_i b_i \doteq 1-\rho_{i_0},$$

and, if $\rho_{i_0}\neq 1$, the inverse, V^{-1}, is given by

$$V^{-1}\boldsymbol{e}_i=\boldsymbol{e}_i+\frac{b_i}{1-\rho_{i_0}}\boldsymbol{\lambda}, \quad i=1, \ldots, i_0.$$

This implies that

$$V^{-1}\boldsymbol{\lambda} = \frac{1}{1-\rho_{i_0}}\boldsymbol{\lambda}.$$

These preliminaries serve to define a suitable 'Ljapunov vector' for proving the following theorem which, for H_N, is the necessity part of Theorem 1.1.

Theorem 2.1. *For system H_{i_0}, $1 \leqslant i_0 \leqslant N$, if the Markov chain (S, X) is positive recurrent, then $\rho_{i_0} < 1$ and*

$$\lambda_i \bar{w}_{i_0} < \pi_i d_i (1-\rho_{i_0}), \quad 1 \leqslant i \leqslant i_0,$$

where

$$\bar{w}_{i_0} = \sum_{i=1}^{N} \pi_i \sum_{j=1}^{N} r_{ij} w_{ij} + \sum_{i=i_0+1}^{N} \pi_i b_i d_i.$$

Furthermore, in case of positive recurrence,

$$\sum_{x} \pi(i, \boldsymbol{x}) E[\min(x_i, D_i)] = \lambda_i \frac{1}{1-\rho_{i_0}} \bar{w}_{i_0}, \quad i \leqslant i_0, \tag{2.1}$$

where $(\pi(i, \boldsymbol{x}))$ denotes the stationary distribution of $(S, \boldsymbol{X})$.

Proof. Fix i_0 and let

$$f(\boldsymbol{x}) = \begin{cases} V^{-1}\boldsymbol{x}, & \rho_{i_0} \neq 1, \\ \boldsymbol{x}, & \rho_{i_0} = 1, \end{cases}$$

for all $\boldsymbol{x} \in \mathbb{R}^{i_0}$.

Let P denote a probability measure corresponding to some fixed starting conditions. Then

$$\frac{1}{n} E[f(\boldsymbol{X}(n)) - f(\boldsymbol{X}(0))] = \frac{1}{n} \sum_{m=0}^{n-1} \sum_{(i,\,\boldsymbol{x})} P(S(m) = i, \boldsymbol{X}(m) = \boldsymbol{x}) \boldsymbol{L}(i, \boldsymbol{x}),$$

where

$$\boldsymbol{L}(i, \boldsymbol{x}) = E[f(\boldsymbol{X}(1)) - f(\boldsymbol{X}(0)) \mid S(0) = i, \boldsymbol{X}(0) = \boldsymbol{x}]$$

$$= \begin{cases} \sum_{j=1}^{N} r_{ij} V^{-1} \boldsymbol{\alpha}(i, j; \boldsymbol{x}), & \rho_{i_0} \neq 1, \\ \sum_{j=1}^{N} r_{ij} \boldsymbol{\alpha}(i, j; \boldsymbol{x}), & \rho_{i_0} = 1. \end{cases}$$

First consider the case that $\rho_{i_0} \neq 1$. The preliminaries on the drift vectors $\boldsymbol{\alpha}(\cdot)$ and the matrix V^{-1} yield

$$\boldsymbol{L}(i, \boldsymbol{x}) = \frac{1}{1-\rho_{i_0}} \sum_{j=1}^{N} r_{ij}(w_{ij} + d_i b_i)\boldsymbol{\lambda}, \quad i > i_0,$$

$$L(i, \boldsymbol{x}) = \frac{1}{1-\rho_{i_0}} \sum_{j=1}^{N} r_{ij} w_{ij} \boldsymbol{\lambda} - d_i(x_i) \boldsymbol{e}_i, \quad i \leqslant i_0 ,$$

where one uses the trivial relationship

$$V^{-1} \boldsymbol{v}_i = \boldsymbol{e}_i, \quad i \leqslant i_0 .$$

This implies the equation

$$\frac{1}{n} E[f(\boldsymbol{X}(n)) - f(\boldsymbol{X}(0))]$$

$$= \frac{1}{1-\rho_{i_0}} \frac{1}{n} \sum_{m=0}^{n-1} \left\{ \sum_{i=1}^{N} P(S(m)=i) \sum_{j=1}^{N} r_{ij} w_{ij} + \sum_{i>i_0} P(S(m)=i) d_i b_i \right\} \boldsymbol{\lambda}$$

$$- \frac{1}{n} \sum_{m=0}^{n-1} \sum_{i \leqslant i_0} \sum_{\boldsymbol{x}} P(S(m)=i, \boldsymbol{X}(m)=\boldsymbol{x}) d_i(x_i) \boldsymbol{e}_i . \tag{2.2}$$

Suppose the chain is positive recurrent. Then a simple argument shows that the left-hand side of the above equation tends to $\mathbf{0}$. To see this, it suffices to show that the sequence

$$\left\{ \frac{1}{n} (\boldsymbol{X}(n) - \boldsymbol{X}(0)) \right\}$$

tends to $\mathbf{0}$ in distribution and is uniformly integrable, both componentwise. The former is directly implied by positive recurrence. Considering the component i,

$$\left| \frac{1}{n} (X_i(n) - X_i(0)) \right| \leqslant \frac{1}{n} \sum_{m=0}^{n-1} |X_i(m+1) - X_i(m)| ,$$

and since the increment $X_i(m+1) - X_i(m)$ is obtained by subtracting the number of services in station i during the mth visit of some station from the number of arrivals to station i during that visit and the walk thereafter, it is not difficult to show that there exists an i.i.d. sequence of nonnegative integrable random variables $Y_0, Y_1, \ldots$ such that, stochastically,

$$\frac{1}{n} \sum_{m=0}^{n-1} |X_i(m+1) - X_i(m)| \leqslant \frac{1}{n} \sum_{m=0}^{n-1} Y_m .$$

The sequence $\{(1/n) \sum_{m=0}^{n-1} Y_m\}$ is well known to be integrable, and hence the desired uniform integrability follows. Thus, the right-hand side of (2.1) tends to 0, componentwise, yielding the relationships

$$\sum_{\boldsymbol{x}} \pi(i, \boldsymbol{x}) d_i(x_i) = \lambda_i \frac{1}{1-\rho_{i_0}} \bar{w}_{i_0}, \quad i \leqslant i_0 .$$

Hence, ρ_{i_0} cannot exceed 1, and since $d_i(x_i) \leqslant d_i$ for all $i, \boldsymbol{x}$, positive recurrence implies that $\rho_{i_0} < 1$ and $\lambda_i \bar{w}_{i_0} < \pi_i d_i (1-\rho_{i_0})$, provided that the case $\rho_{i_0} = 1$ can also be excluded. Let $\rho_{i_0} = 1$. Then

$$L(i, \boldsymbol{x}) = \begin{cases} \left(\sum_{j=1}^{N} r_{ij} w_{ij} + d_i b_i\right)\boldsymbol{\lambda}, & i > i_0, \\ \sum_{j=1}^{N} r_{ij} w_{ij} \boldsymbol{\lambda} - d_i(x_i) \boldsymbol{v}_i, & i \leqslant i_0, \end{cases}$$

and thus

$$\frac{1}{n} E[f(\boldsymbol{X}(n)) - f(\boldsymbol{X}(0))]$$
$$= \frac{1}{n} \sum_{m=0}^{n-1} \left\{ \sum_{i=0}^{N} P(S(m) = i) \sum_{j=1}^{N} r_{ij} w_{ij} + \sum_{i > i_0} P(S(m) = i) d_i b_i \right\} \boldsymbol{\lambda}$$
$$- \frac{1}{n} \sum_{m=0}^{n-1} \sum_{i \leqslant i_0} \sum_{\boldsymbol{x}} P(S(m) = i, \boldsymbol{X}(m) = \boldsymbol{x}) d_i(x_i) \boldsymbol{v}_i .$$

Again, if the chain is positive recurrent, the left-hand side of this equation tends to $\mathbf{0}$, and one obtains the equation

$$\bar{w}_{i_0} \boldsymbol{\lambda} = \sum_{i \leqslant i_0} \sum_{\boldsymbol{x}} \pi(i, \boldsymbol{x}) d_i(x_i) \boldsymbol{v}_i .$$

Now $\boldsymbol{v}_i = \boldsymbol{e}_i - b_i \boldsymbol{\lambda}$, and letting

$$c_i = \sum_{\boldsymbol{x}} \pi(i, \boldsymbol{x}) d_i(x_i), \quad i \leqslant i_0,$$

it follows that

$$\left(\bar{w}_{i_0} + \sum_{i \leqslant i_0} c_i b_i\right) \boldsymbol{\lambda} = \sum_{i \leqslant i_0} c_i \boldsymbol{e}_i ,$$

or, equivalently,

$$\lambda_i \left(\bar{w}_{i_0} + \sum_{i \leqslant i_0} c_i b_i\right) = c_i, \quad i \leqslant i_0 .$$

Multiplying with b_i and summing up yields

$$\rho_{i_0} \left(\bar{w}_{i_0} + \sum_{i \leqslant i_0} c_i b_i\right) = \sum_{i \leqslant i_0} c_i b_i ,$$

which is incompatible with the assumption that $\rho_{i_0} = 1$ (recall that $\bar{w}_{i_0} > 0$ for all i_0). The theorem is proved. □

Remark. Consider the recurrent case. The above proof shows that the sequence

$$\left\{\frac{1}{n}\sum_{m=0}^{n-1} E[X(m+1)-X(m)]\right\}$$

tends to $\mathbf{0}$ as n tends to infinity. Trivially, so does the same sequence linearly transformed by V^{-1}. The transform brings the criterion to light that ρ_{i_0} be less than 1, and it yields the relationships (2.1). The transform has been introduced at an early stage for purposes of reference in the next section. Instead, one can also start as in the case $\rho_{i_0}=1$ and take limits. This yields the relationship

$$\sum_{(i,\,x)} \pi(i,x)(\bar{w}_{i_0}\boldsymbol{\lambda} - d_i(x_i)\boldsymbol{v}_i) = \mathbf{0}\,.$$

This says that the steady-state mean drift is equal to $\mathbf{0}$. Transforming this relationship by V^{-1} yields (2.1) and thus the criterion that ρ_{i_0} be less than 1.

The idea of using a linear transform in a similar context can also be found in [5].

3. Proof of sufficiency

The sufficiency part of Theorem 1.1 is obtained by putting $i_0=N$ in the following theorem, which is about system H_{i_0} as defined in the previous section. The notation of Section 2 is in force.

Theorem 3.1. *For system H_{i_0}, $1\leqslant i_0\leqslant N$, the conditions of Theorem 2.1, i.e. the inequalities*

$$\lambda_i\bar{w}_{i_0} < \pi_i d_i(1-\rho_{i_0}),\quad 1\leqslant i\leqslant i_0\,,$$

are sufficient for (S, X) to be positive recurrent.

Proof. Fix i_0. The definition of H_{i_0} depends on the original numbering of the stations of system H. Assume that the stations 1 through i_0 are renumbered from 1 to i_0 in such a way that the inequalities

$$\frac{\lambda_1}{\pi_1 d_1}\leqslant\frac{\lambda_2}{\pi_2 d_2}\leqslant\cdots\leqslant\frac{\lambda_{i_0}}{\pi_{i_0}d_{i_0}}$$

hold. It follows that if the conditions of Theorem 2.1 are true for H_{i_0}, they are true for H_s, $1\leqslant s\leqslant i_0$. To see this, check $s=i_0-1$ to get for $i\leqslant i_0-1$,

$$\begin{aligned}\lambda_i\bar{w}_{i_0-1} &= \lambda_i\bar{w}_{i_0}+\lambda_i\pi_{i_0}b_{i_0}d_{i_0}\\ &< \pi_i d_i(1-\rho_{i_0})+\lambda_i\pi_{i_0}b_{i_0}d_{i_0}\\ &= \pi_i d_i(1-\rho_{i_0-1})+\pi_i d_i\lambda_{i_0}b_{i_0}+\lambda_i\pi_{i_0}b_{i_0}d_{i_0}\\ &= \pi_i d_i(1-\rho_{i_0-1})-b_{i_0}\pi_i d_i\pi_{i_0}d_{i_0}\left(\frac{\lambda_{i_0}}{\pi_{i_0}d_{i_0}}-\frac{\lambda_i}{\pi_i d_i}\right)\\ &\leqslant \pi_i d_i(1-\rho_{i_0-1})\,.\end{aligned}$$

Hence, in view of (2.2), and noting that f has nonnegative components on the states of the chain, when $\rho_{i_0} < 1$, one has

$$\mathbf{0} < \liminf_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n-1} \sum_{i=1}^{s} \sum_{\boldsymbol{x}} P(S(m)=i, \boldsymbol{X}(m)=\boldsymbol{x})(d_i - d_i(x_i))\boldsymbol{e}_i ,$$

$s=1, \ldots, i_0$, where $(S, \boldsymbol{X})$ denotes the chain corresponding to system H_s. The proof is now by induction. For $s=1$, the above inequality boils down to

$$\mathbf{0} < \liminf_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n-1} \sum_{x_1=0}^{\infty} P(S(m)=1, X_1(m)=x_1)(d_1 - d_1(x_1)) .$$

But $0 \leqslant d_1 - d_1(x_1) < \varepsilon$ for any given positive ε and large enough x_1, and hence

$$\mathbf{0} < \liminf_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n-1} \sum_{x_1 \leqslant K_1(\varepsilon)} P(S(m)=1, X_1(m)=x_1)(d_1 - d_1(x_1))$$

for ε small enough and some finite $K_1(\varepsilon)$. The right-hand side involves finitely many states, only, and hence would be equal to zero if the chain were null-recurrent or transient. Thus positive recurrence is established for H_1.

Assume next that $(S, \boldsymbol{X})$ is positive recurrent for H_{s-1}, where s, $1 < s \leqslant i_0$, is fixed. Consider $(S, \boldsymbol{X})$ for H_s. For given $\varepsilon > 0$ let $K_s(\varepsilon)$ be such that $0 \leqslant d_s - d_s(x_s) < \varepsilon$ for $x_s > K_s(\varepsilon)$. Then

$$\frac{1}{n} \sum_{m=1}^{n-1} \sum_{\boldsymbol{x}: x_s > K_s(\varepsilon)} P(S(m)=s, \boldsymbol{X}(m)=\boldsymbol{x})(d_s - d_s(x_s)) < \varepsilon .$$

Thus, for ε small enough and some finite $K_s(\varepsilon)$,

$$\mathbf{0} < \liminf_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n-1} \sum_{\boldsymbol{x}: x_s \leqslant K_s(\varepsilon)} P(S(m)=s, \boldsymbol{X}(m)=\boldsymbol{x})(d_s - d_s(x_s)) .$$

From the latter double sum split off the summation over the $\boldsymbol{x}$ such that $x_i > K$ for some $i < s$, where K is a suitable constant. This summation is dominated by the expression

$$d_s \frac{1}{n} \sum_{m=1}^{n-1} P(S(m)=s, X_i(m) > K \text{ for some } i \in \{1, \ldots, s-1\}) ,$$

which is a quantity associated with the 'projection' $\{(S(m), X_1(m), \ldots, X_{s-1}(m))\}$ of the process $\{S(m), X_1(m), \ldots, X_s(m)\}$ defined for H_s. This projection should behave no worse than the process $(S, \boldsymbol{X})$ defined for H_{s-1}, and this can indeed be seen by way of an obvious coupling of $(S, \boldsymbol{X})$ for H_s and $(S, \boldsymbol{X})$ for H_{s-1}. One constructs a path of $(S, \boldsymbol{X})$ for H_s and identifies its 'projection' (as above) with an associated path segment for H_{s-1} until the first time when the server in H_s comes to station s and performs less than the D_s-distributed number of services there. At the next step one takes the server to the same station for both processes, but has to add more arrivals to the various stations for the system H_{s-1}, namely those that would arrive during the corresponding dummy services. Then one continues in an obvious way such that, for all n, the $X_i(n)$, $1 \leqslant i \leqslant s-1$, for H_s do not exceed those for H_{s-1}. The properties of the Poisson arrival process make this legitimate. As a consequence, since $(S, \boldsymbol{X})$ is positive recurrent for H_{s-1}, the above split-off summation can be made arbitrarily small, say less than ε again, for sufficiently large K. It follows that, for ε small enough and $K_s(\varepsilon)$, K large enough,

$$\mathbf{0} < \liminf_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n-1} \sum_{x: x_i \leqslant K, 1 \leqslant i \leqslant s-1, x_s \leqslant K_s(\varepsilon)} P(S(m)=s, \boldsymbol{X}(m)=\boldsymbol{x})(d_s - d_s(x_s)),$$

and, since only finitely many states are involved, this implies positive recurrence for the $(S, \boldsymbol{X})$-process associated with H_s. Theorem 3.1 is proved. □

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